Abstract

In this paper an interesting recursive relation is figured out for the Mobius function. The main idea is to represent Lambert coefficients of the given function in a some recursive form. In this paper will be shown an importance of \((1 + x)(1 + x + x^2)\ldots(1 + x + \ldots + x^n)\) product coefficients for the Lambert series.
1 Notations and Well Known Facts

The following notations are used in this paper.

- Lambert series \( \sum_k a_k z^k - 1 \) are denoted as \( L(z) \).
- Mobius function for the natural \( n \) is denoted as \( \mu(n) \).
- The sum \( 1 + x + .. + x^n \) is denoted as \( \lambda_n(x) \).
- The product \( (1 + x)(1 + x + x^2) .. (1 + x + .. + x^n) \) is denoted as \( \psi_n(x) \). So \( \psi_n(x) = \lambda_1(x) .. \lambda_n(x) \).
- For the \( \psi_n(x) \) \( k \)-th order coefficient the notation \( \alpha_k^n \) is used. So \( \psi_n(x) = \sum_{k \leq n(n+1)} \alpha_k^n x^k \).

The following relations are not proved in this paper.

1. \( L(z) = \sum_k \frac{a_k}{z^k - 1} = \sum_k \frac{\mu_d a_d}{z^d} \)
2. \( \sum_{d|n} \mu(d) = 0 \) for \( n > 1 \) and 1 for the \( n=1 \)
3. For the Mobius function \( \sum_k \frac{\mu_k}{z^k - 1} = \sum_k \frac{\mu_k}{z^k} = \frac{1}{2} \) (*)

2 From the Lambert series to recursive polynomial relations

Let consider the following series which is close to Lambert series:

\[ \Phi_n(z) = \sum_{k \geq n} \frac{a_k}{z^k - 1} \]

\( \Phi_n(z) \) allows calculation of Lambert transformation coefficients \( a_k \) easily using the following relation:

\[ \lim_{z \to \infty} \Phi_n(z) z^n = a_n. \text{ (**)} \]

It is obvious that

\[ \Phi_n(z) - \Phi_{n-1}(z) = \frac{a_{n-1}}{z^{n-1} - 1} \]

or

\[ \Phi_n(z) = \Phi_{n-1}(z) + \frac{a_{n-1}}{z^{n-1} - 1} \]

Let consider a few samples of \( \Phi_n(z) \) for the Mobius function:

\( \Phi_1(z) = \sum_k \frac{\mu_k}{z^k - 1} = \frac{1}{2} \) from the (*)

\( \Phi_1(z) = \lim_{z \to \infty} \Phi_1(z) z^1 = 1 \) from the (**)

\( \Phi_2(z) = \frac{1}{2} - \frac{\mu_1}{z-1} = \frac{1}{2} - \frac{1}{z-1} = \frac{z-1-2}{z(z-1)} = -\frac{1}{z(z-1)} \)

\( \Phi_2(z) = \lim_{z \to \infty} \Phi_2(z) z^2 = -1 \)
\( \Phi_3(z) = -\frac{1}{z^{-1}} - \frac{\mu_2}{z^{-1}} - \frac{1}{z^{-1}} - \frac{1}{z^{-1}(z+1)} = \frac{-1(z+1)^{-1} - (z-1)z}{z^{-1}(z+1)} = \frac{-1}{z^{-1}(z+1)} \)

\( \mu_3 = \lim_{z \to \infty} \Phi_3(z)z^3 = -1 \)

\( \Phi_4(z) = \frac{-1}{z(z-1)(z+1)} - \frac{\mu_3}{z^2-1} = \frac{-1}{z(z-1)(z+1)} - \frac{1}{z^2+z+1} = \frac{-z^2-z-1+z^2+z}{z(z-1)(z+1)(z^2+z+1)} = \frac{-1}{z^{-1}(z^2+z+1)} \)

\( \mu_4 = \lim_{z \to \infty} \Phi_4(z)z^4 = 0 \)

Using this recursive method we can calculate \( \Phi_n(z) \) and \( \mu_n \) for any \( n \). Let’s investigate the structure of the \( \Phi_n(z) \) function. If we continue recursion above we will see that

\( \Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)(z^n + z^{n-1} + \ldots + 1)} - \frac{\mu_n}{(z-1)z^{n-1}(z)} \)

The order of \( M_n(z) \) is calculated below in this paper.

Let’s rewrite the last relation using \( \lambda_n(z) \) instead of \( (z^n + z^{n-1} + \ldots + 1) \).

\( \Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)\lambda_n(z)} - \frac{\mu_n}{z(z-1)\lambda_{n-1}(z)} \)

and replacing \( \lambda_1(z) \ldots \lambda_n(z) \) by \( \psi_n(z) \)

\( \Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)\psi_{n-1}(z)} - \frac{\mu_n}{z(z-1)\psi_{n-1}(z)} \)

The last relation can be rewritten in the following way

\( \Phi_{n+1}(z) = \frac{M_n(z)\lambda_{n-1}(z) - \mu_n z \psi_{n-2}(z)}{z(z-1)\psi_{n-1}(z)} \)

Now if

\( \Phi_{n+1}(z) = \frac{M_{n+1}(z)}{Q_{n+1}(z)} \)

then

\( M_{n+1}(z) = M_n(z)\lambda_{n-1}(z) - \mu_n z \psi_{n-2}(z) \)

and

\( Q_{n+1}(z) = z(z-1)\psi_{n-1}(z) \)

And it is a good time to clarify the order of \( M_n(z) \) and to calculate \( \mu_n \). To do this we can use (*)

\( \lim_{z \to \infty} \Phi_n(z)z^n = \mu_n \)

from where we can conclude that \( \Phi_n(z) \) has order less than \(-n\). It is obvious that \( \mu_n \) is 0 if the order of \( \Phi_n(z) \) less than \(-n\) and it differs from 0 only when the order of \( \Phi_n(z) \) is equal \(-n\). Now if we will take into account that

\( \Phi_n(z)z^n = \frac{M_n(z)z^n}{Q_n(z)} = \mu_n \quad z \to \infty \)
and the order of \( Q_n(z) \) is equal \( 1 + 1 + 1 + \ldots + (n - 2) = 2 + \frac{(n-2)(n-1)}{2} \). It is clear then that the order of \( M_n(z) \) is less or equal order(\( Q_n(z) \)) - \( n = 2 + \frac{(n-2)(n-1)}{2} - n = \frac{(n-2)(n-3)}{2} \) and \( \mu_n \) is the \( \frac{(n-2)(n-3)}{2} \)-th coefficient of \( M_n(z) \) or 0 if the order of \( M_n(z) \) is less than \( \frac{(n-2)(n-3)}{2} \).

So finally the goal of this section is reached and we have polynomial recursion for the \( M_n(z) \) the \( \frac{(n-2)(n-3)}{2} \)-th coefficient of which is a Mobius function:

\[
\begin{align*}
M_1(z) &= 1; M_2(z) = -1; M_3(z) = -1; M_4(z) = -1; \\
M_{n+1}(z) &= M_n(z)\lambda_{n-1}(z) - \mu_n\psi_{n-2}(z) \quad (***)
\end{align*}
\]

where \( \mu_n \) is a \( \frac{(n-2)(n-3)}{2} \)-th coefficient of \( M_n(z) \) if there is such \( z \) or 0. And we found that the order of \( M_n(z) \) is less or equal \( \frac{(n-2)(n-3)}{2} \). We need to remember that this rule works for the \( n \geq 4 \).

3 From the recursive polynomials to 2-variable recurrency

The goal of this section is to avoid variable \( z \) and get the regular recursion. Let's denote the order of \( M_n(z) \) as \( \nu(n) \). From previous section we know that the \( \nu(n) \leq \frac{(n-2)(n-3)}{2} \). In other words

\[
M_n(z) = \sum_{0 \leq k \leq \nu(n)} a_n^k z^k.
\]

We can do the same for the \( n+1 \)

\[
M_{n+1}(z) = \sum_{0 \leq k \leq \nu(n+1)} a_{n+1}^k z^k.
\]

To simplify calculations we assume that the order of \( M_n(z) \) is \( \frac{(n-2)(n-3)}{2} \). All the missing coefficients we will assume equal to 0. So we can say that \( \nu(n) = \frac{(n-2)(n-3)}{2} \). Now let use (***)

\[
M_{n+1}(z) = \sum_{0 \leq k \leq \nu(n+1)} a_{n+1}^k z^k = \sum_{0 \leq k \leq \nu(n)} a_n^k z^k \lambda_{n-1}(z) - \sum_{0 \leq k \leq \nu(n)} a_n^k z^k = \sum_{0 \leq q \leq \nu(n-2)} a_n^q z^q = \sum_{0 \leq q \leq \nu(n-2)} a_n^q z^q
\]

or

\[
\sum_{0 \leq k \leq \nu(n+1)} a_{n+1}^k z^k = \sum_{0 \leq k \leq \nu(n)} a_n^k z^k - \sum_{0 \leq q \leq \nu(n-2)} a_n^q z^q
\]

where \( \alpha_m \) is the \( m \)-th order coefficient of the \( \psi_n(z) \) Lets write here recursive relations for each order coefficients:

- \( z=0 \): \( a_{n+1}^0 = a_n^0 = \ldots = a_3^0 = -1 \)
- \( z=1 \): \( a_{n+1}^1 = a_n^0 + a_n^1 - \alpha_{n-2}^0 \nu(n) \)
- \( z=2 \): \( a_{n+1}^2 = a_n^0 + a_n^2 + a_n^2 - \alpha_{n-2}^0 \nu(n) \)
After more detailed investigation we see that there are 3 possible kind of coefficients $a_m^n$ for the given $n$:

- The case $m \leq n - 1$ for which:
  \[
  a_{n+1}^m = \sum_{0 \leq k \leq m} a_n^k - a_n^{\nu(n)} \frac{m-1}{2} \alpha_{n-2}
  \] (1)

- The case $n - 1 < m \leq \nu(n)$ for which:
  \[
  a_{n+1}^m = \sum_{m-n \leq k \leq m} a_n^k - a_n^{\nu(n)} \alpha_{n-2}
  \] (2)

- and the case $\nu(n) < m$ for which:
  \[
  a_{n+1}^m = \sum_{m-n \leq k \leq \nu(n)} a_n^k - a_n^{\nu(n)} \alpha_{n-2}
  \] (3)

with an assumption that $a_m^n = 0$ when $m > \frac{n(n+1)}{2}$ or when $m < 0$. We need to remember that $a_{n+1}^0 = -1$ which is obvious. Now the goal of this section is reached. But the formulas we have are not suitable for understanding. Simplification is done in a next section.

### 4 Simplification of recurrency

Lets consider the difference $a_{n+1}^m - a_{n+1}^{m-1}$. We have 4 important cases:

1. $m \leq n - 1$
   It is obviously follows from the (1):
   \[
   a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^{\nu(n)} (\alpha_{n-2} - \alpha_{n-2})
   \]

2. $n - 1 < m \leq \nu(n)$
   From the (2):
   \[
   a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^{m-n} - a_n^{\nu(n)} (\alpha_{n-2} - \alpha_{n-2})
   \]

3. $\nu(n) < m$
   From the (3):
   \[
   a_{n+1}^m - a_{n+1}^{m-1} = -a_n^{m-n} - a_n^{\nu(n)} (\alpha_{n-2} - \alpha_{n-2})
   \]

4. $m = n - 1 + 1 = n$
   This is a corner case and to calculate the difference the (1) and (2) are used:
   \[
   a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^0 - a_n^{\nu(n)} (\alpha_{n-2} - \alpha_{n-2})
   \]

It is easy to see that the point 4 it is the same as 2. We can combine these points in a one equation:
\[
a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^{m-n} - a_\nu(n) \alpha_{n-2}^{m-1} - a_{n-2}^{m-2} \quad (****)
\]

with an assumption that \( a_n^m = 0 \) if \( m < 0 \) or \( m > \nu(n) = \frac{(n-2)(n-3)}{2} \). To complete this rule we need to remember that:

\[
a_0^0 = -1
\]

and that this rule works for the \( n \geq 5 \). For the \( n=4 \) case the \( \nu(4) = 1 \) so from the (***) we conclude that: \( a_1^1 = -1 \) and \( a_1^2 = 0 \).

So the goal of this section and the goal of this paper is reached. By summarizing this we have for Mobius function:

\[
\mu(n) = \alpha_n^{\nu(n)}
\]

where \( a_n^\nu \) satisfies the (****) recurrence.