Recursive Formula Related To The Mobius Function

Gevorg Hmayakyan

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Abstract

In this paper an interesting recursive relation is figured out for the Mobius function. The main idea is to represent Lambert coefficients of the given function in a some recursive form. In this paper will be shown an importance of $(1+x)(1+x+x^2)..(1+x+..+x^n)$ product coefficients for the Lambert series.

1 Notations and Well Known Facts

The following notations are used in this paper.

- Lambert series $\sum_k \frac{a_k}{z^k-1}$ are denoted as $L(z)$.
- Mobius function for the natural n is denoted as $\mu(n)$.
- The sum $(1+x+..+x^n)$ is denoted as $\lambda_n(x)$.
- The product $(1+x)(1+x+x^2)..(1+x+..+x^n)$ is denoted as $\psi_n(x)$. So $\psi_n(x) = \lambda_1(x)...\lambda_n(x)$.
- For the $\psi_n(x)$ k-th order coefficient the notation α_n^k is used. So $\psi_n(x) = \sum_{k \leq \frac{n(n+1)}{2}} \alpha_n^k x^k$

The following relations are not proved in this paper.

1. $L(z) = \sum_k \frac{a_k}{z^k-1} = \sum_k \frac{\sum_{d|k} a_d}{z^k}$
2. $\sum_{d|n} \mu(d) = 0$ $n > 1$ and 1 for the n=1
3. For the Mobius function

$$\sum_k \frac{\mu_k}{z^k-1} = \sum_k \frac{\sum_{d|k} \mu_d}{z^k} = \frac{1}{z} \text{ (*)}$$

2 From the Lambert series to recursive polynomial relations

Lets consider the following series which is close to Lambert series:

$$\Phi_n(z) = \sum_{k \geq n} \frac{a_k}{z^k-1}$$

$\Phi_n(z)$ allows calculation of Lambert transformation coefficients a_k easily using the following relation:

$$\lim_{z \to \infty} \Phi_n(z) z^n = a_n. \text{ (**)}$$

It is obvious that

$$\Phi_n(z) - \Phi_{n-1}(z) = \frac{a_{n-1}}{z^{n-1}-1}$$

or

$$\Phi_n(z) = \Phi_{n-1}(z) + \frac{a_{n-1}}{z^{n-1}-1}$$

Lets consider a few samples of $\Phi_n(z)$ for the Mobius function:
$\Phi_1(z) = \sum_k \frac{\mu_k}{z^k-1} = \frac{1}{z}$ from the (*)
$\mu_1 = \lim_{z \to \infty} \Phi_1(z) z^1 = 1$ from the (**)

$\Phi_2(z) = \frac{1}{z} - \frac{\mu_1}{z-1} = \frac{1}{z} - \frac{1}{z-1} = \frac{z-1-z}{z(z-1)} = -\frac{1}{z(z-1)}$
$\mu_2 = \lim_{z \to \infty} \Phi_2(z) z^2 = -1$

$\Phi_3(z) = -\frac{1}{z(z-1)} - \frac{\mu_2}{z^2-1} = -\frac{1}{z(z-1)} - \frac{-1}{(z-1)(z+1)} = \frac{-1(z+1)-(-1)z}{z(z-1)(z+1)} = \frac{-1}{z(z-1)(z+1)}$
$\mu_3 = \lim_{z\to\infty} \Phi_3(z)z^3 = -1$

$\Phi_4(z) = \frac{-1}{z(z-1)(z+1)} - \frac{\mu_3}{z^3-1} = \frac{-1}{z(z-1)(z+1)} - \frac{-1}{(z-1)(z^2+z+1)} = \frac{-z^2-z-1+z^2+z}{z(z-1)(z+1)(z^2+z+1)} = \frac{-1}{z(z-1)(z+1)(z^2+z+1)}$
$\mu_4 = \lim_{z\to\infty} \Phi_4(z)z^4 = 0$

Using this recursive method we can calculate $\Phi_n(z)$ and μ_n for any n. Lets investigate the structure of the $\Phi_n(z)$ function. If we continue recursion above we will see that

$$\Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)(z+1)..(z^{n-2}+z^{n-1}+..+1)} - \frac{\mu_n}{(z-1)(z^{n-1}+z^{n-2}+..+1)}$$

The order of $M_n(z)$ is calculated below in this paper.

Lets rewrite the last relation using $\lambda_n(z)$ instead of $(z^n + z^{n-1} + .. + 1)$.

$$\Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)\lambda_1(z)...\lambda_{n-2}(z)} - \frac{\mu_n}{(z-1)\lambda_{n-1}(z)}$$

and replacing $\lambda_1(z)...\lambda_n(z)$ by $\psi_n(z)$

$$\Phi_{n+1}(z) = \frac{M_n(z)}{z(z-1)\psi_{n-2}(z)} - \frac{\mu_n}{(z-1)\lambda_{n-1}(z)}$$

The last relation can be rewritten in the following way

$$\Phi_{n+1}(z) = \frac{M_n(z)\lambda_{n-1}(z)-\mu_n z\psi_{n-2}(z)}{z(z-1)\psi_{n-1}(z)}$$

Now if

$$\Phi_{n+1}(z) = \frac{M_{n+1}(z)}{Q_{n+1}(z)}$$

then

$$M_{n+1}(z) = M_n(z)\lambda_{n-1}(z) - \mu_n z\psi_{n-2}(z)$$

and

$$Q_{n+1}(z) = z(z-1)\psi_{n-1}(z)$$

And it is a good time to clarify the order of $M_n(z)$ and to calculate μ_n. To do this we can use (*)

$$\lim_{z\to\infty} \Phi_n(z)z^n = \mu_n.$$

from where we can conclude that $\Phi_n(z)$ has order less than (-n). It is obvious that μ_n is 0 if the order of $\Phi_n(z)$ less than (-n) and it differs from 0 only when the order of $\Phi_n(z)$ is equal (-n). Now if we will take into account that

$$\Phi_n(z)z^n = \frac{M_n(z)z^n}{Q_n(z)} = \mu_n \; z \to \infty$$

and the order of $Q_n(z)$ is equal $1+1+1+...+(n-2) = 2+\frac{(n-2)(n-1)}{2}$. It is clear then that the order of $M_n(z)$ is less or equal order$(Q_n(z)) - n = 2+\frac{(n-2)(n-1)}{2} - n = \frac{(n-2)(n-3)}{2}$ and μ_n is the $\frac{(n-2)(n-3)}{2}$-th coefficient of $M_n(z)$ or 0 if the order of $M_n(z)$ is less than $\frac{(n-2)(n-3)}{2}$.

So finally the goal of this section is reached and we have polynomial recursion for the $M_n(z)$ the $\frac{(n-2)(n-3)}{2}$-th coefficient of which is a Mobius function:

$$\boxed{M_1(z) = 1;\ M_2(z) = -1;\ M_3(z) = -1;\ M_4(z) = -1;}$$

$$\boxed{M_{n+1}(z) = M_n(z)\lambda_{n-1}(z) - \mu_n z\psi_{n-2}(z)\ (***)}$$

where μ_n is a $\frac{(n-2)(n-3)}{2}$-th coefficient of $M_n(z)$ if there is such z or 0. And we found that the order of $M_n(z)$ is less or equal $\frac{(n-2)(n-3)}{2}$. We need to remember that this rule works for the $n \geq 4$.

3 From the recursive polynomials to 2-variable recurrency

The goal of this section is to avoid variable z and get the regular recursion. Lets denote the order of $M_n(z)$ as a $\nu(n)$. From previous section we know that the $\nu(n) \leq \frac{(n-2)(n-3)}{2}$. In other words

$$M_n(z) = \sum_{0\leq k\leq \nu(n)} a_n^k z^k.$$

We can do the same for the n+1

$$M_{n+1}(z) = \sum_{0\leq k\leq \nu(n+1)} a_{n+1}^k z^k.$$

To simplify calculations we assume that the order of $M_n(z)$ is $\frac{(n-2)(n-3)}{2}$. All the missing coefficients we will assume equal to 0. So we can say that $\nu(n) = \frac{(n-2)(n-3)}{2}$. Now let use (***).

$$M_{n+1}(z) = \sum_{0\leq k\leq \nu(n+1)} a_{n+1}^k z^k = \sum_{0\leq k\leq \nu(n)} a_n^k z^k \lambda_{n-1}(z) - a_n^{\nu(n)}\psi_{n-2}(z)z =$$
$$\sum_{0\leq q\leq n-1}\sum_{0\leq k\leq \nu(n)} a_n^k z^{k+q} - a_n^{\nu(n)}\psi_{n-2}(z)z =$$
$$\sum_{0\leq q\leq n-1}\sum_{0\leq k\leq \nu(n)} a_n^k z^{k+q} - a_n^{\nu(n)}\sum_{0\leq r\leq \nu(n-2)} \alpha_{n-2}^r z^{r+1}$$

or

$$\sum_{0\leq k\leq \nu(n+1)} a_{n+1}^k z^k =$$
$$\sum_{0\leq k\leq \nu(n)}\sum_{0\leq q\leq n-1} a_n^k z^{k+q} - a_n^{\nu(n)}\sum_{0\leq r\leq \nu(n-2)} \alpha_{n-2}^r z^{r+1}$$

where α_n^m is the m-th order coefficient of the $\psi_n(z)$ Lets write here recursive relations for each order coefficients:

- z=0: $a_{n+1}^0 = a_n^0 = ... = a_3^0 = -1$
- z=1: $a_{n+1}^1 = a_n^0 + a_n^1 - \alpha_{n-2}^0 a_n^{\nu(n)}$
- z=2: $a_{n+1}^2 = a_n^0 + a_n^1 + a_n^2 - \alpha_{n-2}^1 a_n^{\nu(n)}$

After more detailed investigation we see that there are 3 possible kind of coefficients a_n^m for the given n:

- The case $m \leq n-1$ for which:

$$\boxed{a_{n+1}^m = \sum_{0 \leq k \leq m} a_n^k - a_n^{\nu(n)} \alpha_{n-2}^{m-1}} \quad (1)$$

- The case $n-1 < m \leq \nu(n)$ for which:

$$\boxed{a_{n+1}^m = \sum_{m-n \leq k \leq m} a_n^k - a_n^{\nu(n)} \alpha_{n-2}^{m-1}} \quad (2)$$

- and the case $\nu(n) < m$ for which:

$$\boxed{a_{n+1}^m = \sum_{m-n \leq k \leq \nu(n)} a_n^k - a_n^{\nu(n)} \alpha_{n-2}^{m-1}} \quad (3)$$

with an assumption that $\alpha_n^m = 0$ when $m > \frac{n(n+1)}{2}$ or when $m < 0$. We need to remember that $a_{n+1}^0 = -1$ which is obvious. Now the goal of this section is reached. But the formulas we have are not suitable for understanding. Simplification is done in a next section.

4 Simplification of recurrency

Lets consider the difference $a_{n+1}^m - a_{n+1}^{m-1}$. We have 4 important cases:

1. $m \leq n-1$
   It is obviously follows from the (1):

$$a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^{\nu(n)}(\alpha_{n-2}^{m-1} - \alpha_{n-2}^{m-2})$$

2. $n-1 < m \leq \nu(n)$
   From the (2):

$$a_{n+1}^m - a_{n+1}^{m-1} = a_n^m - a_n^{m-n} - a_n^{\nu(n)}(\alpha_{n-2}^{m-1} - \alpha_{n-2}^{m-2})$$

3. $\nu(n) < m$
   From the (3):

$$a_{n+1}^m - a_{n+1}^{m-1} = -a_n^{m-n} - a_n^{\nu(n)}(\alpha_{n-2}^{m-1} - \alpha_{n-2}^{m-2})$$

4. $m = n-1+1 = n$
   This is a corner case and to calculate the difference the (1) and (2) are used:

$$a_{n+1}^m - a_{n+1}^{m-1} = a_n^n - a_n^0 - a_n^{\nu(n)}(\alpha_{n-2}^{m-1} - \alpha_{n-2}^{m-2})$$

It is easy to see that the point 4 it is the same as 2. We can combine these points in a one equation:

$$\boxed{a_{n+1}^{m} - a_{n+1}^{m-1} = a_n^m - a_n^{m-n} - a_n^{\nu(n)}(\alpha_{n-2}^{m-1} - \alpha_{n-2}^{m-2})\ (****)}$$

with an assumption that $a_n^m = 0$ if $m < 0$ or $m > \nu(n) = \frac{(n-2)(n-3)}{2}$. To complete this rule we need to remember that:

$$\boxed{a_n^0 = -1}$$

and that this rule works for the $n \geq 5$. For the n=4 case the $\nu(4) = 1$ so from the (***) we conclude that: $a_4^0 = -1$ and $a_4^1 = 0$.
So the goal of this section and the goal of this paper is reached. By summarizing this we have for Mobius function:

$$\boxed{\mu(n) = a_n^{\nu(n)}}$$

where a_n^m satisfies the (****) recurrency.